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Note

A Coloring Problem Related to the Erdős–Faber–Lovász Conjecture

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Let $f(n)$ be the maximum chromatic number among the graphs which can be covered by n copies of K_n. It is shown that $\lim_{n\to\infty} f(n)/n^{3/2}=1$.

The famous conjecture of Erdős, Faber, and Lovász (see, e.g., [2]) states that if the edge set of a graph G can be covered with n copies of K_n, the complete graph on n vertices, such that any two of those K_n share at most one vertex, then the chromatic number $\chi(G)$ of G is just n. The best upper bound so far has been proved by W. I. Chang and E. Lawler in [1].

In the present note we determine the asymptotical maximum of the chromatic number in the case when the intersection of those K_n is not restricted.

THEOREM 1. *The chromatic number of any graph whose edge set can be covered with n copies of K_n is at most $n^{3/2}$, and this bound is asymptotically best possible.*

In fact, this result is a particular case of the following more general statement.

THEOREM 2. *Let G be a graph whose edge set can be covered with k complete subgraphs, each having at most n vertices. Then $\chi(G)\leqslant nk^{1/2}$. Moreover, if k tends to infinity with n, and $k=o(n^2)$, then there exists a sequence of such graphs G_k with $\chi(G_k)\geqslant(1-o(1))\,nk^{1/2}$.*

Proof. The union of k complete graphs of orders at most n has at most $\frac{1}{2}k(n^2-n)$ edges. Thus, the number of vertices of degree at least $nk^{1/2}$ in

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any induced subgraph $G' \leqslant G$ is less than $nk^{1/2}$, implying that every G' contains a vertex of degree less than $nk^{1/2}$. Consequently, $\chi(G) \leqslant nk^{1/2}$.

To obtain a lower bound, let p be the largest prime power not exceeding $k^{1/2}$. It is well known that $p/k^{1/2} = 1 - o(1)$. Then the projective plane $PG(p)$ of order p has $k - o(k)$ points and the same number of lines. Replace each point v_i by a set V_i of cardinality $\lfloor n/(p+1) \rfloor = (1 - o(1))n/k^{1/2}$. Then each line L_j defines a set $F_j = \bigcup_{v_i \in L_j} V_i$ of at most n elements, and by the incidence axioms of $PG(p)$ each pair in $V = \bigcup_i V_i$ is contained in some F_j. Thus, the complete graph with vertex set V of $(1 - o(1))\, nk^{1/2}$ elements has an edge covering with at most k complete subgraphs of order at most n (induced by those F_j).

One can see that Theorems 1 and 2 remain valid if the copies of K_n are assumed to share at most $n^{1/2}$ and $n/k^{1/2}$ vertices, respectively.

References

1. W. I. Chang and E. Lawler, Edge coloring of hypergraphs and conjecture of Erdős, Faber, Lovász, *Combinatorica* **8** (1988), 293–295.
2. P. Erdős, On the combinatorial problems which I would most like to see solved, *Combinatorica* **1** (1981), 25–42.